

# Picture Invariance in Quantum Optics

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We clarify the controversy over the coherent-state (CS) versus the number-state (NS) pictures in quantum optics. The NS picture is equivalent to the CS picture, as long as the phases  $\phi$  in the laser fields are randomly distributed, as Mølmer argues [Phys. Rev. A **55**, 3195 (1997)]. However, the claim by Rudolph and Sanders [Phys. Rev. Lett. **87**, 077903 (2001)] has a few gaps. First, they make an assumption that is not necessarily true in the calculation of a density operator involved with a two-mode squeezed state. We show that there exists entanglement in the density operator without defying the assumption that phases are randomly distributed. Moreover, using a concept of picture-invariance, we argue that it is not that criteria for quantum teleportation are not satisfied. We discuss an analogy between the controversy on the CS versus NS pictures to that on the heliocentric versus geocentric pictures.

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## I. INTRODUCTION

Besides continuing controversies over its implication on our understanding of the physical world [1, 2, 3, 4, 5], quantum mechanics is still revealing its hidden aspects, now in a form of quantum information processing [6]. The coherent-state (CS) picture has been successful in describing quantum optical phenomena [9, 10, 11]. However, the fact that a picture is successful does not exclude the possibility of other pictures. Indeed, the CS picture is not the only choice [13].

Quantum teleportation (QT) [14] is an interesting ingredient of quantum information processing: Utilizing the nonlocality of Einstein-Podolsky-Rosen pairs of [1, 2, 3] quantum bits (qubits), QT enables us to do a task whose result is equivalent to actual transport of qubits. Recently, a quantum optical experiment was performed by Furusawa *et al.* [15] to demonstrate a continuous-variable version [16, 17] of QT. However, Rudolph and Sanders (RS) [18] criticized the experiment in Ref. 15 by extending the argument of Mølmer [13]. RS were then criticized by van Enk and Fuchs [19]. Sanders *et al.* again supported Ref. 18 in a subsequent work [20]. Besides those, a few authors have joined the controversy [21, 22, 23]. In fact, Refs. 21 and 22 give criticisms that are quite similar to ours. However, our viewpoint is not the same as theirs [21, 22], especially in that we discuss a concept of 'picture' in connection with the controversy.

The purpose of this paper is to clarify the controversy. The number-state (NS) picture is equivalent to the CS picture in the sense that the NS picture can also explain quantum optical phenomena, as Mølmer argues [13]. However, the arguments of RS have a few gaps. First there is a technical gap in their interpretation of the experiment, as pointed out by Yuen [24]. Next, although we cannot fix our picture, we can define certain properties that are invariant for transformation between various pictures, including the CS and the NS pictures. (This is in analogy with the fact that although we cannot fix our reference frame, we have some facts, e.g., 'two objects are in contact', that are invariant to transformation

of the reference frame in classical mechanics.) Using the concept of picture-invariance, we can see that the usual interpretation using the CS picture is meaningful enough. This paper is organized as follows: First, we describe the NS picture. Then, we point out the technical gap in the argument of RS. Next, we introduce picture-invariance. Then, we argue, with the picture-invariance, how RS's claim is not valid. Then, we discuss an analogy between the controversy on CS versus NS pictures to that on the heliocentric versus geocentric pictures, and we conclude.

## II. NUMBER STATE PICTURE

Let us discuss a fact about an absolute phase  $\phi$  of the laser field: Even if we can perform a real phase measurement, the absolute value of the phase has no meaning [25]. The meaning becomes clear when we consider an analogy between the phase value and spatial coordinate value. Even if we can measure the distance  $\Delta x = x_1 - x_2$  between two points  $x_1$  and  $x_2$  in space, it is meaningless to argue what absolute value should be assigned to a point, e.g.,  $x_1$ . In other words, we can assign any value to a point. In this sense only, we cannot measure or determine the absolute phase of a laser field (Proposition-0).

However, an actual laser field is not in a pure coherent state  $|\alpha e^{i\phi}\rangle = \exp(-\alpha^2/2) \sum_{n=0}^{\infty} (\alpha^n e^{in\phi} / \sqrt{n!}) |n\rangle$ , but in a mixed state

$$\rho_L = \int_0^{2\pi} \frac{d\phi}{2\pi} |\alpha e^{i\phi}\rangle \langle \alpha e^{i\phi}| \quad (1)$$

$$= e^{-\alpha^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} |n\rangle \langle n|. \quad (2)$$

The reason is the following: Values of the phase  $\phi$  in laser fields are randomly distributed (Proposition-1) [13]. As a principle of quantum mechanics, however, we cannot distinguish two different decompositions of the same density operator [6]. Therefore, we may well choose any decomposition for the laser field  $\rho_L$  in describing quantum optical phenomena involved with  $\rho_L$ . For example,

it can be either a decomposition in Eq. (1) (CS picture) or that in Eq. (2) (NS picture). As long as Proposition-1 and quantum mechanics are correct, the above argument by Mølmer [13] cannot be incorrect.

### III. A GAP IN THE ARGUMENTS OF RUDOLPH AND SANDERS

Let us be reminded of the argument by RS [18]. If a pure coherent state  $|\alpha e^{i\phi}\rangle$  is used to pump nonlinear crystals, the two-mode squeezed state

$$|\eta e^{i\phi}\rangle = \sqrt{1-\eta^2} \sum_{n=0}^{\infty} \eta^n e^{in\phi} |nn\rangle \quad (3)$$

is generated. Here, the phase  $\phi$  of the pumping state is transcribed into that of a generated state. Since the phase-randomized state  $\rho_L$  is used to pump, the generated state  $\rho_S$  is also phase randomized,

$$\rho_S = \int_0^{2\pi} \frac{d\phi}{2\pi} |\eta e^{i\phi}\rangle \langle \eta e^{i\phi}| = (1-\eta^2) \sum_{n=0}^{\infty} \eta^{2n} |nn\rangle \langle nn|. \quad (4)$$

A reasonable definition of separability is that as long as a density operator  $\rho$  is separable in any decomposition, it is separable [6]. Thus, the phase-randomized two-mode squeezed state  $\rho_S$  is separable. RS claim that there is no entanglement in places where entanglement is supposed to be in normal QT. In their criticism for RS, van Enk and Fuchs [19] make a claim that there is entanglement in the state. However, they assume an experiment that measures the phase of a laser field. The problem is that the experiment has not yet been realized although the possibility is not excluded in principle. van Enk and Fuchs claim that whether we perform the experiment or not, the fact that there exist entanglement in the state of Eq. (12) of Ref. 19 does not change. However, this is not the case. Let us consider a similar illustrating example where Alice and Bob share a state

$$|\Psi\rangle = \frac{1}{2}(|0\rangle|\phi^+\rangle + |1\rangle|\phi^-\rangle + |2\rangle|\psi^+\rangle + |3\rangle|\psi^-\rangle) \quad (5)$$

Here,  $|0\rangle, |1\rangle, |2\rangle$ , and  $|3\rangle$  are normalized and orthogonal states, and, for example,  $|0\rangle|\phi^+\rangle$  denotes  $|0\rangle_A|\phi^+\rangle_{AB}$  where  $A$  and  $B$  denote Alice and Bob, respectively. Alice and Bob are assumed to be remotely separated as usual. Bell states are given by  $|\phi^\pm\rangle_{AB} = (1/\sqrt{2})(|0\rangle_A|0\rangle_B \pm |1\rangle_A|1\rangle_B)$  and  $|\psi^\pm\rangle_{AB} = (1/\sqrt{2})(|0\rangle_A|1\rangle_B \pm |1\rangle_A|0\rangle_B)$ . A measurement that distinguishes the four states  $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$  would reduce the state  $|\Psi\rangle$  to one of the Bell states, and the outcome of the measurement would identify which Bell state it is. Therefore, if Alice can perform the measurement, it amounts to their sharing a Bell state. However, if Alice has no way of performing the measurement by a reason, e.g., because she lost the first qubit, the second and the

third qubits they share no longer constitute a Bell state, but constitute a purely separable state described by

$$\frac{1}{4}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|) \quad (6)$$

$$= \frac{1}{4}I = \frac{1}{4}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|). \quad (7)$$

Here,  $I$  is an identity operator. One might argue that an objective property like entanglement must not depend on subjective knowledge. However, it is one of the most interesting facts in quantum information that entanglement depends on subjective knowledge. In the same way, there is no entanglement in the state of Eq. (12) of Ref. 19 before the phase measurement is realized.

What we show here is that even without the phase measurement there remains entanglement in the usual two-mode squeezed state. The problem in RS's argument is that in their interpretation of the experiment, they make an assumption that is not necessarily true: A laser used to pump a nonlinear crystal is reset each time when a pair of photon pulses is generated (Prescription-1). However, *Prescription-1 is not followed in a real experiment*. Once the pumping laser is turned on, it is used for a while. However, a coherent state  $|\alpha e^{i\phi}\rangle$  can be split into copies of the same phase in  $m$  different modes,

$$|\alpha e^{i\phi}\rangle \rightarrow \left| \frac{\alpha}{\sqrt{m}} e^{i\phi} \right\rangle^{\otimes m}, \quad (8)$$

where  $m$  is a positive integer and  $\alpha, \beta$  are real numbers. For the case of spatial modes, the result in Eq. (8) can be obtained by simple beam splitters and phase-shifters, as is well known [26]. For the case of temporal modes, the result in Eq. (8) is obtained if we keep using the same laser beam for a duration of time [19]. In either case, each state  $|\frac{\alpha}{\sqrt{m}} e^{i\phi}\rangle$  in  $m$  different modes is used to pump a nonlinear crystal. Then, the phases of the two-mode squeezed state are the same. The corresponding density operator is, therefore, not  $\rho_S$ , but

$$\rho_T = \int_0^{2\pi} \frac{d\phi}{2\pi} \left| \frac{\eta}{\sqrt{m}} e^{i\phi} \right\rangle^{\otimes m} \left\langle \frac{\eta}{\sqrt{m}} e^{i\phi} \right\rangle^{\otimes m}. \quad (9)$$

Note that the states  $\rho_S$  and  $\rho_T$  are different. Although the state  $\rho_S$  has no entanglement, the state  $\rho_T$  has entanglement.

### IV. PICTURE INVARIANCE AND EXISTENCE OF ENTANGLEMENT

Before showing the existence of entanglement in the state  $\rho_T$ , let us discuss picture-invariance. Picture-invariant quantities or properties are those that do not depend on any particular decomposition of a certain density operator. One example is the density operator itself. Another example is existence of entanglement or non-separability. Recall the definition of non-separability: If a density operator is not separable for any decomposition

of the density operator, it is non-separable [6]. Measures of entanglement, e.g., entanglement of formation [6], are also picture-invariant. Note that we may well use any particular decomposition in estimating picture-invariant quantities or properties. In other words, whatever picture we use, we get the same result for picture-invariant quantities or properties.

Now let us show that the state  $\rho_T$  has entanglement. That may be done by proving that a measure of entanglement for the state  $\rho_T$  has a non-vanishing value. However, our method is to show that the state  $\rho_T$  can violate Bell's inequality [2, 3]. It is known [27, 28] that the two-mode squeezed state  $|\eta e^{i\phi}\rangle^{\otimes m}$  can violate the Bell's inequality by measuring even and odd parities if all instruments share a single reference beam and if Prescription-1 is not followed. Here, the problem is that a beam (the reference beam), besides the beams corresponding to the state  $\rho_T$ , is shared by Alice and Bob. Therefore, even if the Bell's inequality is violated, one cannot say that it is solely due to the state  $\rho_T$ . However, we consider an experiment where Alice and Bob do not share any beam except for the ones corresponding to the state  $\rho_T$ . Even in this case, however, we can see that it can violate Bell's inequality: Let us assume that the Bell's inequality is violated in a specific case where  $\phi = p$ ,  $\alpha = a$ , and  $\beta = b$  in Eq. (6) of Ref. 28. We repeat many times the experiment for the state  $\rho_T$ . In each experiment,  $m$  pairs of photon pulses are measured.  $m$  is large enough to give statistical confirmation. The phase is randomly given in each experiment because Alice and Bob do not share a reference beam. When the phases are far from the desired values,  $p$ ,  $a$ , and  $b$ , Bell's inequality may not be violated. However, when the phases happen to be close to the desired ones, Bell's inequality is violated. The result is that Bell's inequality is violated for  $m$  pairs with a fixed success probability  $P$ . One might say that in this case, we cannot say that Bell's inequality is violated because we have selected certain samples that violate Bell's inequality. However, that is not the case because the number  $m$  can be large enough with a fixed success probability  $P$ . Assume that we simply post-select samples that violate Bell's inequality in a row from a larger sample that does not violate Bell's inequality. In this case, the success probability  $P$  should exponentially decrease with the size  $m$  of the sample that violate Bell's inequality. In the above case, however, the success probability  $P$  is fixed. Therefore, we can say that the state  $\rho_T$  is nonlocal and, thus, has entanglement.

We note that nonlocality of a state is a picture-invariant property. Whatever picture we choose, we will get the same result for a given state. Therefore, the above result obtained by the CS picture is valid. Here, a picture is just a mathematical tool to get a picture-invariant result.

Let us now criticize RS's claim. Let us consider the three criteria of QT [18]. (a) The state to be teleported is unknown to a sender, Alice, and a receiver, Bob, and is supplied by an actual third party, Victor. (b) Alice

and Bob share only a nonlocal entangled resource and a classical channel through which Alice transmits her measurement results to Bob. (c) Entanglement should be a verifiable resource. RS argue that none of the three criteria is satisfied in the experiment by Furusawa *et al.* [15]. As we have seen already, however, the criterion (c) is satisfied in the experiment.

Next, let us consider the criterion (b). RS say that Alice and Bob have extra entanglement via a pumping laser they share in the experiment; thus, criterion (b) is not satisfied. In the NS picture, apparently there is entanglement between the shared laser and light beams split from it. However, in this case, the entanglement is not a genuine one in the following sense: Let us consider again the example of a mixed state of qubit-pairs whose density operator is described by either Eq. (6) or (7). In Eq. (7), however, the density operator is interpreted as a mixture of product states while in Eq. (6), the density operator is interpreted as a mixture of highly entangled states. Separability is a picture-invariant property. In other words, it is not reasonable that a property like separability depends on the picture we choose. Therefore, we can say that there is no entanglement in the density operator  $(1/4)I$  because the density operator is separable in a picture. In the same way, it is reasonable to say that entanglement, between the shared laser and light beams split from it, in the NS picture is not a real entanglement. One can choose whatever picture he/she wants, but that is the case only when a picture is used as a calculational tool. To summarize, because the state of the system, except for the one corresponding to  $\rho_T$ , is separable according to usual definition that is picture-invariant, we can say that the criterion (b) is satisfied in the experiment.

Let us interpret Smolin's discussion [23] on criterion (a) with the concept of picture invariance. In the CS picture, the states are separable. However, Victor rotates his beam with a phase that is randomly chosen by him by using a phase shifter. The random phase is what Alice and Bob do not know, which is a picture-invariant quantity. What if we use the NS picture? Can the information on Victor's action be transferred via (artificial) entanglement in the NS picture? (RS argues that there are many correlations in the two beams of Eq. (2) of Ref. 18. Also, they implied that these correlations can be utilized in exchanging information. However, what the correlations mean is just that the states have *the same* phase in the CS picture before Victor's rotating action.) However, a physical fact, like Alice and Bob not being able to get information on Victor's rotating action via the shared pumping laser, must be picture-invariant. This means that even if there are entanglements in the NS picture, after careful considerations, those entanglements will turn out not to be useful for transferring information from Victor to Alice and Bob. Therefore, we can say that the criterion (a) is also satisfied in the experiment.

## V. DISCUSSION AND CONCLUSION

It is interesting that the controversy on the CS versus NS pictures has an analogy to the old one on the heliocentric versus geocentric pictures on motions of planets in the solar system. Both pictures work as a tool for calculating observed phenomena. We cannot say that an interpretation is correct while the other one is incorrect. However, as Ocam proposed, what is simple or convenient should be preferred. In this sense, the CS picture is better than the NS picture: Picture-invariant properties like separability manifest themselves in the CS picture while picture-invariant properties often disguise themselves in the NS pictures. We are not committing the partition-ensemble-fallacy [29]: We are not talking about picture-specific ones, but talking about picture-invariant ones.

In conclusion, the NS picture is equivalent to the CS picture as Mølmer argues [13], as long as Proposition-1 is correct. However, claim by RS [18] has a few gaps. First, they make an assumption that is not necessarily

true in the calculation of a density operator involved with two-mode squeezed state. We showed that there exists entanglement in the density operator without defying the Proposition-1. Moreover, using the concept of picture-invariance that we introduced, we argued that two criteria for QT are also satisfied. Then, we discussed an analogy between the controversy on the CS versus NS pictures to that on the geocentric versus heliocentric pictures. We argued why we were not committing the partition-ensemble-fallacy [29].

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